Formulas used in RogerHub's Final Grade Calculator

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1 Introduction

This document explains the formulas used in RogerHub's Final Grade Calculator.

2 Grade required on the final exam

The original Final Grade Calculator was created in 2009. Given the following inputs, it determines the minimum percentage grade required on the final exam.

- c = Current overall grade, excluding the final exam
- d = Desired overall grade
- w = Weight of the final exam

With this, we can express f, the final exam score, as:

$$f \geq \frac{d - (1 - w)c}{w}$$

This calculator implicitly assumes that the only undetermined category is the final exam. Inputs are converted to be fractions between 0 and 1 (however, values of c and d outside this range are also permitted). Output values are formatted as percentages and rounded to the nearest hundredth of a percent.

3 Extended Calculators

In the summer of 2013, several additional calculators were added to the Final Grade Calculator. They were chosen from the most frequently asked questions in the comments section.

3.1 Overall grade after final exam

A new calculator was created for "Already took the final. What is my overall grade?". Given the following inputs, it calculates the resulting overall grade including the final exam.

- c = Overall grade before the final exam
- f = Score on final exam
- w = Weight of the final exam

The answer is:

$$(1 - w)c + wf$$

As you guessed, this is simply the result of isolating d in the formula for the original calculator.

3.2 Grade required on the final test

A new calculator was created for "My final counts as a test. What do I need to get?". Given the following inputs, it determines the minimum percentage grade required on the final exam that counts as part of the regular "test category".

- c = Current overall grade
- d = Desired overall grade
- w = Weight of the test category
- n = Number of tests taken
- a = Average score in the test category
- t = The final test is worth this number of test-equivalent units

Once again, this calculator assumes that the only undetermined category is the test category. It also assumes that all tests are equal in weight, with the exception of the final exam which is worth some multiple of a standard test-equivalent unit.

If tests are not equally weighted, the calculator can still be used in a roundabout way. The variables n and t are always used in conjunction, such that only the ratio $\frac{t}{n+t}$ needs to be correct. Replacing n with the number of total possible test points available and t with the number of test points on the final test would also yield an accurate result.

To reach the answer, we define auxiliary variable τ:

$$\text{let } \tau = \begin{cases} 0, & \text{if } w = 1 \\ \dfrac{c - wa}{1 - w}, & \text{if } w < 1 \end{cases}$$

The variable τ represents the average grade of the non-test categories. These categories are not affected by the final test (represented below as f). It then follows that:

$$d \leq (1 - w)\tau + w\frac{an + ft}{n + t}$$

Isolating f, we get:

$$f \geq \frac{(d - (1 - w)\tau)(n + t) - wan}{wt}$$

When $w < 1$, this simplifies to:

$$f \geq \frac{(d - c + wa)(n + t) - wan}{wt}$$

When $w = 1$, the expression becomes:

$$f \geq \frac{d(n + t) - an}{t}$$

This takes a form similar to the original calculator's formula, because if tests are the only remaining category, then the final test can be treated as its own undetermined category with weight $\frac{t}{n+t}$.

3.3 Grade required on a multi-part final exam

A new calculator was created for "There are 2+ parts to my final. What do I have to get on them?". This calculator is made specifically for multi-day final exams, where some of the parts are already graded, but none of the parts have been included into the overall grade yet. Given the following inputs, it determines the minimum average percentage grade required on the ungraded parts of the final exam.

- c = Current overall grade, not including any part of the multi-part final
- d = Desired overall grade
- w = Total weight of the final
- N = The total number of parts on the final.
- N_C = The number of parts on the final that have been graded.
- w_k = The number of points possible on part k of the final.
- c_k = The number of points scored on part k of the final.

The sizes of sequence w_k and c_k are N and N_C respectively. The solution to the multi-part final problem is to apply the simple final exam formula twice, using the result of the first calculation as the variable d of the second calculation. First, we define:

$$\text{let } W = 1 - \frac{\sum_{i}^{N_C} w_i}{\sum_{i}^{N} w_i} \text{ and } C = \frac{\sum_{i}^{N_C} c_i}{\sum_{i}^{N_C} w_i}$$

These two variables W and C represent the worth of the remaining parts of the final, as a percentage of the whole final, and the weighted average score on the parts of the final that have been taken. With this, we can express f, the average score on the ungraded parts of the final exam, as:

$$f \geq \frac{\frac{d-(1-w)c}{w} - (1-W)C}{W}$$

If either $w = 0$ or $W = 0$, then the requirement becomes infinite (either positve or negative), as expected.

3.4 Weight of the final exam when using a point system

By request, an additional calculator was added to calculate the weight of the final exam when using a point system. Given the following inputs, it determines how much the final exam is worth as a percentage of the entire grade:

- P = Total number of points possible, including the final
- P_F = Number of points the final is worth

With this, we can express w, the weight of the final, as:

$$w = \frac{P_F}{P}$$

3.5 Grade required on the final exam if tests are dropped

A new calculator was created for "My lowest test grade is dropped. What do I need to get?". This calculator handles test forgiveness, in which some fixed number of low test grades are forgotten when determining the overall grade. It also handles the case where the final exam score also counts as a test grade. Given the following inputs, it determines the minimum percentage grade required on the final exam.

- c = Current overall grade, not including test forgiveness or the final exam
- d = Desired overall grade
- w_T = Weight of the test category
- N_T = Number of total tests
- N_D = Number of lowest test grades dropped
- c_T = Average score in the test category
- ℓ_k = Lowest kth test grade
- N_F = The final exam is worth this number of test-equivalent units (can be 0)
- w_F = Weight of the final exam as a separate category (can be 0)

The current overall grade c consists of 3 components:

- The tests that will be dropped from the grade
- The tests that will not be dropped from the grade
- The non-test part of the grade

The eventual overall grade will also consist of 3 components, two of which are preserved from before:

1. The tests that will not be dropped from the grade
2. The non-test part of the grade
3. The final exam

The value of 1. and 2. can be derived from the inputs. Then, we can calculate the effective final exam worth in order to compute an answer.

3.5.1 Non-dropped tests

We can express the average score of the non-dropped tests as:

$$\frac{N_T c_T - \sum_k^{N_D} \ell_k}{N_T - N_D}$$

This score will factor into the final grade as part of the test category, but it will not represent the entire test category if $N_F > 0$. We can say that the non-dropped tests have weight:

$$w_T \frac{N_T - N_D}{N_T - N_D + N_F}$$

To get the total value of the non-dropped tests, we need to multiply these together:

$$\frac{N_T c_T - \sum_k^{N_D} \ell_k}{N_T - N_D} w_T \frac{N_T - N_D}{N_T - N_D + N_F}$$

Let's eliminate the discontinuity at $N_T = N_D$. If this is true, then all test grades are dropped (which is strange, but plausible) and so the value of non-dropped tests should be 0. Note that if the inputs are correct, then $N_T c_T - \sum_k^{N_D} \ell_k$ should be 0 as well. So, we can simplify the above expression to:

$$\left(N_T c_T - \sum_k^{N_D} \ell_k\right) \frac{w_T}{N_T - N_D + N_F}$$

3.5.2 Non-test grades

We do not explicitly ask the user about their non-test grades, but we can infer their value based on the current overall grade. The weight of test grades, as a fraction of the *current* overall grade, is:

$$\frac{w_T}{1 - w_F}$$

Therefore, the average score of the non-test grades should be:

$$\frac{c - \frac{c_T w_T}{1 - w_F}}{1 - \frac{w_T}{1 - w_F}}$$

The non-test grades have weight $(1 - w_T - w_F)$, so we can express the value of the non-test grades as:

$$\frac{c - \frac{c_T w_T}{1 - w_F}}{1 - \frac{w_T}{1 - w_F}} (1 - w_T - w_F)$$

This can be simplified to:

$$\frac{c(1 - w_F) - c_T w_T}{1 - w_T - w_F} (1 - w_T - w_F)$$

If $1 - w_T - w_F = 0$, then there are no non-test categories, and the total value of non-test grades should be 0. So, let's simplify the above expression to:

$$c(1 - w_F) - c_T w_T$$

3.5.3 The final exam

The final exam gets a weight of w_F from the final exam category and a weight of $\frac{w_T N_F}{N_T - N_D + N_F}$ from the test category. This gives us a total weight of:

$$w_F + \frac{w_T N_F}{N_T - N_D + N_F}$$

3.5.4 The formula

First, we express the desired grade d in terms of the 3 components of the eventual overall grade, using f as the final exam score.

$$d \leq \left(N_T c_T - \sum_k^{N_D} \ell_k\right) \frac{w_T}{N_T - N_D + N_F} + c(1 - w_F) - c_T w_T + f\left(w_F + \frac{w_T N_F}{N_T - N_D + N_F}\right)$$

Then, isolating f, we get:

$$f \geq \left(d - \left(N_T c_T - \sum_k^{N_D} \ell_k\right) \frac{w_T}{N_T - N_D + N_F} - (c(1 - w_F) - c_T w_T)\right) / \left(w_F + \frac{w_T N_F}{N_T - N_D + N_F}\right)$$

4 Advanced mode

In 2018, a new mode for "I need something else (advanced mode)" was added to the Final Grade Calculator. This mode calculates the minimum required average score on an arbitrary number of upcoming exams and other assignments while also accounting for a number of unusual circumstances.

4.1 Model

The advanced mode is implemented as a step-by-step questionnaire, in which the answer to each question determines subsequent questions. At its core, the calculator works by determining two numbers:

- $c =$ The current cumulative grade, expressed as the sum of each individual score's *eventual* value.
- $w =$ The total potential value of future scores (exams and other assignments)

At the very end, the advanced mode will prompt the user for d, the desired grade, and then the original Final Grade Calculator formula is used to determine the minimum average score required for all the remaining exams and assignments:

$$f = \frac{d - (1 - w)c}{w}$$

If w is 0, then the calculator will simply report the value of c as the overall grade.

This model assumes no variability in all future scores. In practice, the desired grade can be attained even if some future scores are lower than the requirement, as long as the weighted average of all future scores meets the requirement. This assumption is required in order to produce a useful numeric answer rather than a multidimensional space of possibilities.

4.2 Categories

The advanced mode assumes that the overall grade is computed as the weighted sum of many categories, each of which represents a fixed percentage of the total grade. If this is not the case, then the calculator presents a simple calculator for point-based grading or asks the user to provide additional feedback.

Next, the advanced mode asks for the names and weights of the categories. The names are used for display purposes. The weights must add up to 1.0.

For each category, the advanced mode presents the following choices:

- The whole category has been graded
- The category is empty
- Each score in the category is worth an equal fraction of the category
- Each score in the category is worth some number of points in the category
- Each score in the category is worth some percentage of the category

Depending on the choice, the calculator will present a particular interface for inputting information about the weight and score of each item in the category. The purpose of this stage is to determine the category's contribution to c and w. Graded items contribute their value to c, while ungraded items contribute their potential value to w.

4.3 Anything else?

For each category, the advanced mode allows the inclusion of zero or more exceptional circumstances that affect the category's contribution to c and w. Depending on which of the category structures was chosen in the previous step, some or all of these exceptions may be unavailable.

- Dropping the lowest scores
- Extra credit
- Reattempting a score
- Restoring a missing score
- Final exam also counts toward the category

The advanced mode also presents an option for when the final exam replaces one of the category's scores. However, this situation is equivalent to the combination of two of the other exceptions (with the exception that the final exam is assumed to *always* replace the lowest scores, regardless of whether the final exam score is actually higher than those lowest scores), and so the calculator instructs the user to use those instead.

4.3.1 Dropping the lowest scores

This exception is similar to the "My lowest test grade is dropped. What do I need to get?" calculator, except that the advanced mode also handles cases where not all of the tests have been graded. This exception is only available when the "equal fraction of the category" structure is chosen, since dropping the lowest score is not meaningful with the other choices. The inputs are:

- N = Number of total items in the category
- N_D = Number of low scores to be dropped
- N_R = Number of items in the category that have not been scored yet
- c_k = The kth score, where $1 \le k \le N - N_R$
- ℓ_k = The kth lowest score, where $1 \le k \le \min(N_D, N - N_R)$

After the lowest N_D scores are dropped, the category will contain $N - N_D$ scores. These scores can be divided into 3 categories:

- Fixed: $N_F = \max(N - N_R - N_D, 0)$ scores are already determined and will definitely be included in the category score
- Variable: $N_V = \max(N_R - N_D, 0)$ scores are not determined and will definitely contribute to w (total value of future scores)
- Conditional: $N_C = N - N_D - N_F - N_V$ scores are either fixed or variable, depending on the exact value of the N_R future scores

The category's contribution to c and w need to be adjusted:

- Instead of contributing all $N - N_R$ items to c, the category now only contributes N_F items. Each item in the category now carries more weight, since there are N_D fewer total items. The category score is also adjusted to account for the ℓ_k dropped scores.
- Instead of contributing N_R/N of the category's weight to w, the category now contributes only $N_V/(N - N_D)$ of its weight.

Now, we need to handle the N_C conditional scores. This is simplified by the nonvariability assumption mentioned earlier. We express the N_C conditional scores as a list of tuples, each of which consists of a threshold s_T and a weight s_W. The contribution of each conditional score to c and w depends on whether f, the final answer, is above or below s_T:

- When $f \geq s_T$, the conditional score weight s_W should be added to w (total value of future scores).
- When $f < s_T$, the value $s_T s_W$ should be added to c (current cumulative grade).

The advance mode tries to minimize the value of f. To this end, the conditional score tuples are sorted in ascending order by their thresholds s_T, and then these rules are applied:

- Assuming $f = s_T$, calculate the projected overall grade.
- If the projected overall grade is higher than the goal d, then stop. All previous tuples will contribute their s_W to w, and all remaining tuples (including the currently considered tuple) will contribute $s_T s_W$ toward c.
- If there are more tuples, repeat these rules with the next one. Otherwise, all tuples will contribute their s_W to w.

This strategy aims to incorporate the fewest tuples while still meeting the goal d, which allows the final answer (and the final grade) to be as low as possible.

4.3.2 Extra credit

Extra credit can take many forms. The advanced mode supports extra credit that:

- Adds a fixed amount to a category's score;
- Adds a fixed amount to one score within a category; or
- Adds a fixed amount of points to a points category

All of these cases are interpreted as contributions to c (current cumulative grade).

4.3.3 Reattempting a score

If a known score will be *replaced* by a *unknown* score, then the calculator can adjust c and w to account for this. One important detail about this exception is that the total number of scores within a category does not change. So, the necessary adjustments consist of a decrease in c reflecting the discarded score and an increase in w reflecting the weight of a future score. The exact value of the adjustments depend on how many items are in the category and the value of the discarded score.

4.3.4 Restoring a missing score

If a 0 score will be *replaced* by a *known* score, then the calculator can adjust c to account for this. Like before, this exception does not change the total number of scores within a category. The changes consist of an increase in c reflecting the expected higher score.

4.3.5 Final exam also counts toward the category

If the final exam score will be incorporated into an existing category, then the advanced mode must lower the effective weight of the category. This is equivalent to carving out a new category that contributes 100% of its weight to w, which makes this exception mostly orthogonal and compatible with the others (including dropping the lowest scores).

Determining the exact weight of the new category requires additional input, so the calculator currently only supports this exception when the category structure is equally-weighted scores or when the category uses points. For the former case, it's possible for the final exam to count as a non-integer multiple of a regular score (for example, 1.5 tests grades).